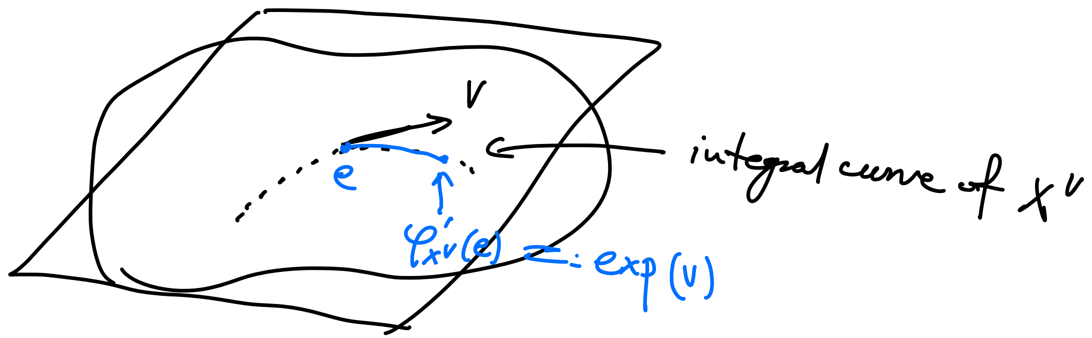
2. Exponential map

<u>Def</u> Given a Lie group G and its Lie algebra $\mathfrak{g}_G$, the exponential map $\exp : \mathfrak{g}_G \to G$ is defined by

$$v \longrightarrow \varphi^1_{X^v}(e) \; \left(= \gamma(1) \right)$$

where γ is the integral curve of X^v starting at e, pointing v.

where $\varphi^1_{X^v}$ is the time-1 map of the 1-par group of diffeo $\{\varphi^t_{X^v}\}_{t\in\mathbb{R}}$ on G generated by X^v.



Basic observations:

- $\exp(s\cdot v) = \varphi^s_{X^v}(e)$ ⟵ rep the 1-par group of diffeo $\{\varphi^t_{X^v}\}_{t\in\mathbb{R}}$ to $\{\varphi^{st}_{X^v}\}_{t\in\mathbb{R}}$ for any $s\in\mathbb{R}$.

- $\exp((s_1+s_2)\cdot v) = \varphi_{X^v}^{s_1+s_2}(e) \underset{\uparrow}{=} \varphi_X^{s_1}(e)\cdot\varphi_X^{s_2}(e) = \exp(s_1 v)\cdot\exp(s_2 v)$

b/c $\{\varphi_{X^v}^s\}$ is a 1-par group of diffeo

$\Rightarrow$ $\exp((s+(-s))v) = \exp(sv)\cdot\exp((-s)v) = e$

which implies $\exp((-s)v) = \exp(sv)^{-1}$, $\forall s \in \mathbb{R}$

- Consider the pushforward of exp:

$$d(\exp)(\underset{\mathfrak{g}\,(\cong \mathbb{R}^{\dim G})}{\underset{\cap}{0}}) : T_0\mathfrak{g} \cong \mathfrak{g} \longrightarrow T_eG \cong \mathfrak{g}$$

computes as follows:

$$d(\exp)(0)(v) = \lim_{t\to 0} \frac{\exp(tv) - \exp(0)}{t}$$

$$= \frac{d}{dt}\Big|_{t=0} \varphi_{X^v}^t(e) \underset{\substack{\uparrow \\ \text{def of integral curve}}}{=} X^v(e) = v$$

$\Rightarrow$ $d(\exp)(0) : \mathfrak{g} \to \mathfrak{g}$ is an identity map (so it is a bijective)

$\Rightarrow$ locally near 0, exp is a diffeomorphism.
constant rank thm

- Given a Lie group homomorphism $\varphi : G \to H$, exp map fits into the following commutative diagram:

$$\begin{tikzcd} \mathfrak{g}_G \arrow[r, "\exp"] \arrow[d, "(d\varphi)(e)"'] \arrow[dr, phantom, "\circlearrowright"] & G \arrow[d, "\varphi"] \\ \mathfrak{g}_H \arrow[r, "\exp"'] & H \end{tikzcd}$$

DIY.

Then if $\varphi_1, \varphi_2 : G \to H$ are Lie group homomorphisms s.t $(d\varphi_1)(e_G) = (d\varphi_2)(e_G)$. Then $\varphi_1 \circ \exp = \varphi_2 \circ \exp$.

Since locally (near 0) exp is a diffeo, $\exists\ U$, a NBH of $e_G \in G$

s.t. $\varphi_1|_U = \varphi_2|_U$.

extra condition

Assume G is connected, then we will show that

$$G = \bigcup_{n\geq 1} \{x_1 \cdots x_n \mid x_i \in U\} \qquad (*)$$

ordered product

$\Rightarrow \varphi_1 = \varphi_2$ since both are homomorphisms.

To prove $(*)$, here is a trick: consider $V = U \cap U^{-1}$ $= \{x^{-1} \mid x \in U\}$.

Then $H := \bigcup_{n\geq 1} \{y_1 \cdots y_n \mid y_i \in V\} \subset \bigcup_{n\geq 1} \{x_1 \cdots x_n \mid x_i \in U\}$ $(\subseteq G)$

is both open and closed.

Open: for any $\sigma \in H$, $\exists\ n_0$ s.t. $\sigma \in \{y_1 \cdots y_{n_0} \mid y_i \in V\}$. Then
$\sigma \cdot V$ is a NBH of σ that is open in G

Closed: $K = G \setminus \bigcup_{\sigma \notin H} \sigma \cdot H$ and each $\sigma \cdot H$ is open. $\square$

To summary, we just proved: if G is connected, then if $\varphi_1, \varphi_2 : G \to H$ are Lie group homomorphism s.t. $(d\varphi_1)(e_G) = (d\varphi_2)(e_G)$, then $\varphi_1 = \varphi_2$.

Rmk This is the first example showing that local (linear approx) info determine the global info on maps.

Thm (famous) Any two simply connected Lie groups are isomorphic (as Lie groups) iff their Lie algebras are isomorphic (as Lie algebras).

$\Uparrow$ The nontrivial step is that given a Lie group G and its Lie algebra $\mathfrak{g}_G$, for any Lie subalgebra $\mathfrak{h} \leq \mathfrak{g}_G$, $\exists$ Lie subgroup $H \leq G$ s.t. $\mathfrak{g}_H = \mathfrak{h}$.

Lie's third thm (called by Serre)

Thm implies $\{$classifying simply connected Lie group$\}$ $\longleftrightarrow$ $\{$classifying Lie algebra$\}$

$\longleftarrow$ Thm above

$\longrightarrow$ Ado-Iwasaw Thm + simply connected

Example (exponential map)

$G = GL(n,\mathbb{R})$. For $v \in \mathfrak{gl}(n,\mathbb{R})$, the left invariant v.f is

$$X^v(A) = A\cdot v \quad \text{(computed earlier)}$$

Then any integral curve $\gamma(t)$ starting at $\mathbb{1} \in GL(n,\mathbb{R})$ by definition is

$$\dot{\gamma}(t) = X^v(\gamma(t)) = \gamma(t)\cdot v \quad \text{with} \quad \gamma(0) = \mathbb{1}$$

and it has a unique solution $\gamma(t) = e^{tv}$ ← this is probably the reason why it's call exp map

Therefore, $\exp : \mathfrak{gl}(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$ $\quad v \mapsto \exp(v) = e^v$

(Recall, e^v means $e^v = \mathbb{1} + v + \frac{v^2}{2} + \cdots$).

One more property of exp map.

Question. what is $\exp(v+w)$? $\quad$ as fcns $e^{v+w} = e^v \cdot e^w$

Prop. $\exp(tv)\cdot\exp(tw) = \exp\left(t(v+w) + o(t^2)\right)$ $\quad \forall\, v, w \in \mathfrak{g}_G$ and sufficiently small $t \in \mathbb{R}$.

Pf Consider

$$\begin{tikzcd}
(-\varepsilon, \varepsilon) \arrow[r] & G \times G \arrow[r, "\text{multiplication}"] & G \\
t \arrow[r, mapsto] \arrow[drr, "\text{a path } \varphi(t)"'] & (\exp(tv), \exp(tw)) \arrow[r, mapsto] & \exp(tv)\cdot\exp(tw) \arrow[d, "\exp^{-1}\ \text{(when } t \text{ is sufficiently small)}"] \\
 & & \mathfrak{g}
\end{tikzcd}$$

$\mathfrak{g}$ ← vector space

$$\varphi(t) := \exp^{-1}\left(\exp(tv)\cdot\exp(tw)\right)$$

In particular $\varphi(0) = \exp^{-1}(e\cdot e) = 0.$

$$\varphi'(t)\Big|_{t=0} = d(\exp^{-1})_0\left(\frac{d}{dt}\Big|_{t=0}\exp(tv) + \frac{d}{dt}\Big|_{t=0}\exp(tw)\right) = v + w$$

$\Longrightarrow$ Taylor expansion in calculus

$$\varphi(t) = 0 + \varphi'(t)\cdot t + \frac{\varphi''(t)}{2}\cdot t^2 + \cdots = (v+w)\cdot t + o(t^2)$$

$\square$

Rmk This formula is one of the key step in the proof of Cartan's closed subgroup Thm (i.e. $H \subseteq G$ closed subset $\Rightarrow$ closed sub-Lie group.)

Rmk Curious about higher order terms ?

$$\exp(v)\cdot\exp(w) = \exp\Big(v+w+\underbrace{\tfrac{1}{2}[v,w]+\tfrac{1}{12}[v,[v,w]]-\tfrac{1}{12}[w,[v,w]]+\cdots}\Big).$$

all higher order terms can be expressed only via bracket.!

This formula is called Baker-Campell-Hausdroff formula.

In particular, if $\mathfrak{g}_G$ is abelian (i.e. $[\,,\,]\equiv 0$), then $\exp(v)\cdot\exp(w) = \exp(v+w)$ on G.

Rmk If $\mathfrak{g}_G$ is abelian, then for $t\in(-\varepsilon,\varepsilon)$, $\exp(tv)\cdot\exp(tw) = \exp(t(v+w)) = \exp(t(w+v)) = \exp(tw)\cdot\exp(tv)$. Therefore, $0\in\mathfrak{g}$ admits a neighborhood V s.t $\operatorname{im}(\exp: V\to G) =: U \subset G$ is abelian. So G is abelian (Why?).

nbhd of e

$\Rightarrow$ G is abelian $\Longleftrightarrow$ $\mathfrak{g}_G$ is abelian (strengthened an earlier result).
(and connected)

3. Calculus on Lie group

Similarly to left invariant vector field, one can define left inv. forms.

Def Given a Lie group G, a left inv k-form on G is a a k-form $\alpha \in \Omega^k(G)$ s.t.

$$(L_g)^*\alpha = \alpha \qquad \forall g \in G.$$

Denote by $\Omega^k_{inv} = \{ \alpha \in \Omega^k(G) \mid \alpha \text{ is a left inv } k\text{-form} \}$

This is a vector space. (compared with the collection of left inv v.f. Ω^k_{inv} does not admit $[\,,\,]$ operation).

However, we have "$\wedge$" wedge.

Consider
$$\Omega^*_{inv}(G) := \bigoplus_{p=0}^{\dim G} \Omega^k_{inv}(G)$$

This forms an algebra under the wedge operation.

$$(L_g)^*(\underset{\underset{\Omega^k_{inv}(G)}{\cap}}{\alpha} \wedge \underset{\underset{\Omega^l_{inv}(G)}{\cap}}{\beta}) = (L_g)^*\alpha \wedge (L_g)^*\beta = \alpha \wedge \beta$$

Let's try some elementary cases.

① $f \in \Omega^0_{inv}(G)$, by def. $(L_g)^* f = f$ i.e.

$$\underset{\uparrow}{f(e)} = (L_g^* f)(e) = f(g) \quad \forall g \in G \Rightarrow f \text{ is a constant function.}$$

function f
evaluated at $e \in G$

② $\alpha \in \Omega^1_{inv}(G)$. For any left inv v.f X on G, consider

$\alpha(X) \in C^\infty(G;\mathbb{R})$, then

$$(L_g)^*(\alpha(X)) = (L_g)^*(\iota_X \alpha) = \iota_{(L_g)^* X}(L_g)^*\alpha$$

This only makes sense if a map is a diffeo.

$$= \iota_{(L_{g^{-1}})_* X}(L_g)^*\alpha = \alpha(X).$$

$\Rightarrow \alpha(X) \in \Omega^0_{inv}(G)$ so $\alpha(X)$ is a constant fcn on G.

Abstractly, we have a linear map

$$\alpha : \underbrace{\{\text{left inv. v.f on } G\}}_{= \mathfrak{g}_G} \longrightarrow \mathbb{R}$$

and if $\alpha(X) = 0$ for any $X \in \mathfrak{g}_G$, then $\alpha = 0$

so $\Omega^1_{inv}(G)$ is the dual space of $\mathfrak{g}_G$, usually denoted by $\mathfrak{g}_G^*$.

③ For $\alpha \in \Omega^1_{inv}(G)$ or equivalently $\alpha \in \mathfrak{g}_G^*$, we have, $X, Y \in \mathfrak{g}_G$

$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$$

b/c $(L_g)^* d = d(L_g)^*$. $\quad d\alpha \in \Omega^2_{inv}(G)$

$$= \underset{\substack{\uparrow \\ \text{b/c } \alpha(Y) \text{ constant}}}{0} - 0 - \alpha([X,Y])$$

This is again a constant fcn b/c $[X,Y] \in \mathfrak{g}_G$.

In general, take

as left inv v.f.

$$\mathfrak{g}_G = \text{span}\{X_1, \dots, X_n\} \qquad X_i \text{ basis.} \quad n = \dim G$$

$$\Omega^1_{inv}(G) = \mathfrak{g}_G^* = \text{span}\{\alpha_1, \dots, \alpha_n\} \qquad \alpha_i \text{ is the dual basis of } X_i.$$

Then $[X_i, X_j] = \sum_{k=1}^n C_{ijk} X_k$ where C_{ijk} are called structure constant of G (w.r.t basis $\{X_1, \dots, X_n\}$).

Then